

# Natural Moves for Knots and Links.

**Abstract.** We propose some natural generalizations of Reidemeister moves that do not increase the number of crossings in the generated diagrams. Experimentations make us conjecture that this class of monotonic moves is complete for computing canonical forms and then deciding isotopy.

## 1. Introduction.

It is a classical result that Reidemeister moves enable to transform any link diagram into any other equivalent one. However, that is not completely satisfactory for computations since the identification of two  $n$  crossings equivalent diagrams may need to generate intermediary diagrams with  $m > n$  crossings. Some works have been done in order to bound this  $m$  at least for the unknotting problem (see [2]).

Here, we investigate another direction : find new types of moves that do not increase the number of crossings. Hence, the possible generated diagrams are in finite number (up to their Gauss codes representations) and we obtain a possible efficient decision algorithm.

We have not yet proved that these moves are complete, that is, enable to transform every pair of equivalent link diagrams in a common one. However, we have already implemented these moves for knots. That enabled to rebuild the tables of prime knots for  $n = 3 \dots 12$  crossings. For larger  $n$ , the computations have to be done.

## 2. Definitions.

First we propose a more general notion of strands.

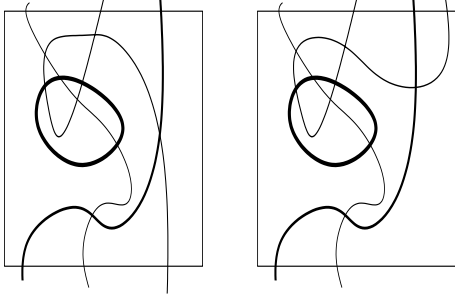
**Definition. (fragment).** Let  $X$  be a link projection. A *fragment* of  $X$  is a continuous part of the curve  $X$ . A fragment is said *close* if it has no extremities. Otherwise, it has two extremities and is said *open*. A collection  $C$  of fragments of  $X$  is said *complete* if every crossing occuring in  $C$  has its two occurrences in  $C$ .

Consider the circular Gauss code  $G$  of  $X$ . One can see a fragment of  $X$  as a part of  $G$ . For instance, the trefoil projection  $X$  with Gauss code  $G = AbCaBc$  has fragments associated to the parts  $AbCaB$  or  $cA$  of  $G$ . The fragment  $bCaBcA$  is closed. The collection  $\{bC, Bc\}$  is complete.

Now, we define a more general notion of braid.

**Definition. (box).** For any  $n, m \geq 0$ , a  $n, m$ -box is a complete collection  $C$  of fragments embedded in a planar square where  $n$  (resp.  $m$ ) extremities are on the top (resp. bottom). Denote  $C^+$  (resp.  $C^-$ ) the list of extremities at the top (resp. bottom) of  $C$ . The *product*  $A.B$  of a  $n, m$ -box  $A$  with a  $m, p$  box  $B$  is obtained by the one-to-one identification of the  $m$  extremities at the bottom of  $A$  with the  $m$  ones at the top of  $B$ , i.e., by  $A^- = B^+$

Here is a 3, 3-box and a 4, 2 box made of 4 fragments (one is closed) :



Observe that every  $n$ -strands braid is a  $n, n$ -box.

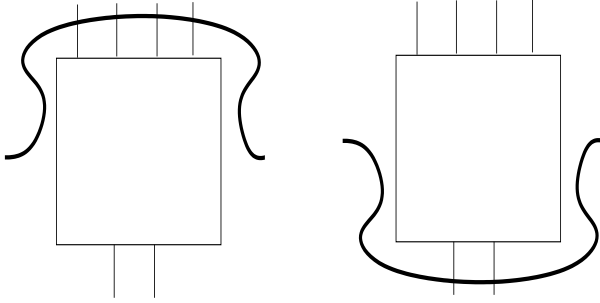
### 3. The moves.

First, we keep the basic Reidemeister moves  $R1$  and  $R2$  but only in the directions that decrease the number of crossings :  $R1^-$ ,  $R2^-$ . We also keep  $R3$  moves since they do not increase the number of crossings.

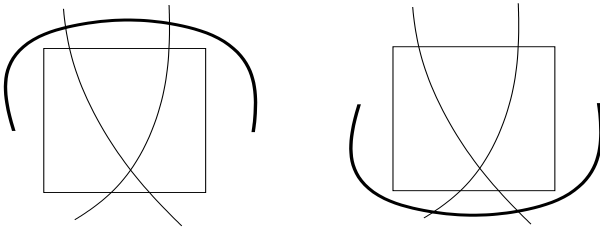
Second, we generalize  $R3$  in two ways : a  $2D$  operation and a  $3D$  operation.

**Definition. (shift).** Let  $C$  be a collection of fragments made of a  $n, m$ -box  $B$  and a fragment  $F$  that crosses over  $B^+$ . The *shift* of  $C$  is obtained by moving the fragment  $F$  over  $B^-$ .

For example :



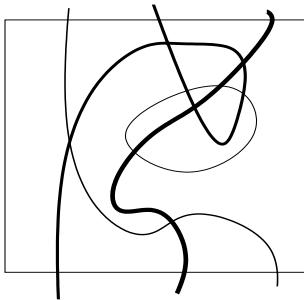
The shift move is a  $2D$  generalization of the  $R3$  move :



Consider  $\Delta_n$ , the Garside's fundamental  $n$ -strands braid (see [1]) and imagine it in  $3D$  as a rotation of  $n$  parallel strands in the space. We do the same for every  $n$  boxes.

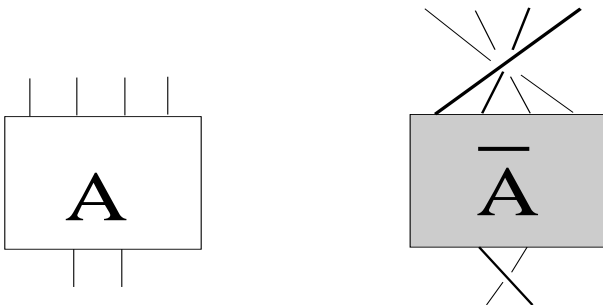
**Definition. (rotation).** The *rotation*  $\overline{C}$  of an  $n, m$ -box  $C$  is obtained by inverting the order of its extremities and its crossings.

The rotation of the first example is :

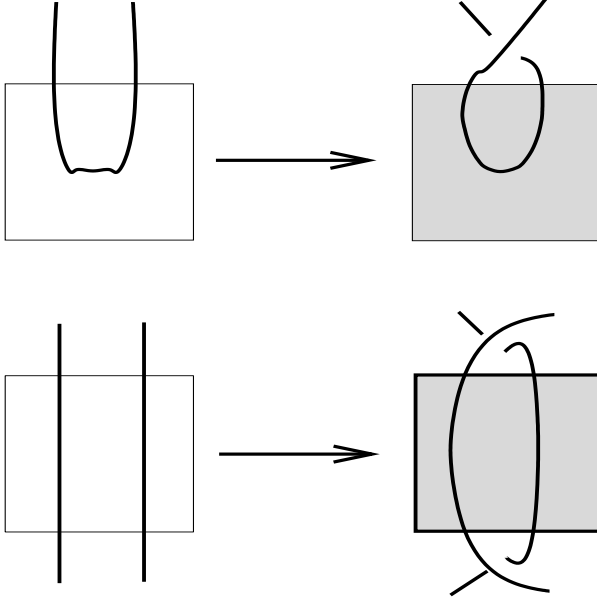


**Definition. (twist).** Let  $A$  be a  $n, m$ -box. The *twist* of  $A$  is obtained by a complete rotation of  $A$  after fixing its extremities. We obtain the  $n, m$  box  $\Delta_n \cdot \overline{A} \cdot \Delta_m^{-1}$ .

For example :



The twist move is a generalization of the  $R1$  and  $R2$  moves :



Now we make the restriction that gives monotonic moves. Of course shift and twist may introduce new crossings. However, we perform then reductions of diagrams by sequences of  $R1^-$ ,  $R2^-$ ,  $R3$  moves.

**Definition. (GENERIC).** A *generic* move consists in the application of a shift or a twist followed by a sequence of  $R1^-$ ,  $R2^-$ ,  $R3$  moves such that a projection  $X$  with  $n$  crossings is transformed in a diagram  $Y$  with  $m \leq n$  crossings.

#### 4. Implementation.

It is not very difficult to implement generic moves on Gauss Codes representations. Moreover, using the combinatorial characterizations of realizable Gauss codes (see [5]), one can decide if some fragments of a diagram can be drawn in a  $n, m$ -box, considering the realizability of the box circled by a single curve.

We have checked generic moves for knots, considering all possible diagrams on  $n = 3 \dots 12$  crossings (using again [5]) and we have obtained reduced knots diagrams (up to mirror images). Computations remain to be done for  $n \geq 13$ . In order to deal with amphicheirality, one has to consider the mirror images of the moves instead of the mirror images of the diagrams.

We have obtained the expected sets of prime knots according to [6]. Moreover, computations are quite fast. For instance, the Perko pair identification is done

by a single generic move from a twist of 3 fragments. Programs and tables are available on :

<http://www2.univ-reunion.fr/~burckel/box.html>

## 5. Questions.

Question 1 : Can one prove that generic moves are complete for knots (links)?

Question  $\bar{1}$  : Can one find two projections  $X, Y$  of equivalent knots (links) that cannot be identified by generic moves ?

Question 2 : Can one generalize generic moves ?

Question 3 : Bounding the number of crossings seems to be quite natural in order to produce effective and efficient algorithms. Are there other quantities that could be bounded in some well orderings?

Question 4 : Some particular twist on 2 fragments is usually called *flype*. The Tait Flyping Conjecture [7] says that flypes enable to decide the isotopy of alternating knots. This important statement has been proved by Kauffman [3], Murasugi [4] and Thistlethwaite [8]. We conjecture that generic moves on 3 fragments enable to decide the isotopy of knots in general.

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